Journal of Computational and Applied Mathematics 234 (2010) 2239–2249

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Determination of precession and dissipation parameters in micromagnetism

I. Cimrák *, V. Melicher

NfaM[2] *Research Group, Department of Mathematical Analysis, Ghent University, Galglaan 2, B-9000 Ghent, Belgium*

ARTICLE INFO

Article history:
Received 19 December 2008
Received in revised form 25 February 2009

Keywords:
Micromagnetism
Parameter estimation
Inverse problems
Sensitivity analysis

ABSTRACT

The precession β and the dissipation parameter α of a ferromagnetic material can be considered microscopically space dependent. Their space distribution is difficult to obtain by direct measurements. In this article we consider an inverse problem, where we aim at recovering α and β from space measurements of the magnetization. The evolution of the magnetization in micromagnetism is governed by the Landau–Lifshitz (LL) equation. We first study the sensitivity of the LL equation. We derive the existence, uniqueness and stability results for the LL equation and the corresponding sensitivity equations. On the basis of the results we analyze the inverse problem. We employ the energy method and we minimize the underlying cost functional by means of the steepest descent method. We derive a convergence result for the proposed algorithm. The presented numerical examples support the theoretical results.

1. Introduction

In micromagnetism, the evolution of the *magnetization* $\mathbf{m}$ around the *effective field* $\mathbf{H}_{\text{eff}}$ is governed by the Landau–Lifshitz (LL) equation

$$\partial_t \mathbf{m} = -\alpha \mathbf{m} \times \mathbf{m} \times \mathbf{H}_{\text{eff}} - \beta \mathbf{m} \times \mathbf{H}_{\text{eff}} \quad \text{in } \Omega, \tag{1}$$

Ω being a bounded domain. The LL equation is accompanied by the homogeneous Neumann boundary condition $\langle \nabla \mathbf{m}, \mathbf{n} \rangle = 0$ on the boundary $\partial\Omega$ of Ω. In general, $\mathbf{H}_{\text{eff}}$ consists of several contributions—e.g. anisotropy, demagnetizing, applied and exchange fields. Exchange field $\mathbf{H}_{\text{ex}}$ generates the highest partial derivatives of $\mathbf{m}$. Therefore we set $\mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{ex}} = \Delta \mathbf{m}$.

The speed of the dissipation of $\mathbf{m}$ is given by *damping* factor α and the rate of precession of $\mathbf{m}$ is given by *precession* parameter β. For non-uniform composite materials the damping and the precession parameters can be space-dependent functions. From a physical point of view it is natural to assume that there exist positive real numbers $\alpha_{\min}$, $\alpha_{\max}$ and $\beta_{\max}$, such that

$$0 < \alpha_{\min} \le \alpha(\mathbf{x}) \le \alpha_{\max}, \qquad |\beta(\mathbf{x})| \le \beta_{\max}. \tag{2}$$

In the rest of the paper we do not write the dependence on space variable explicitly.

Suppose that α and β are known. Then using (1), for any initial state of the magnetization $\mathbf{m}^0$ we can compute the distribution of $\mathbf{m}$ in Ω for any finite time $t = T$. This is the *direct* problem. We are interested in the corresponding *inverse* problem. We aim at the determination of space-dependent functions α and β from the time measurements of the magnetization over the whole domain Ω. We suppose that measured values of the magnetization can be approximated by a function denoted by $\mathbf{m}^*(t)$ belonging to the $L^\infty(\Omega)$ space for any time t. Our aim is thus to find α, β such that the solution $\mathbf{m}(\alpha, \beta)$ to the

* Corresponding author.
E-mail addresses: ivan.cimrak@ugent.be (I. Cimrák), valdemar.melicher@ugent.be (V. Melicher).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.08.081

direct problem will fit the measured values $\mathbf{m}^*(t)$. We will use notation $\mathbf{m}(\alpha, \beta)$ to emphasize that the magnetization is obtained from the direct problem for particular α, β.

A similar problem has been studied in [1,2], where the sensitivity analysis of the LL equation was given for the first time. Only the sensitivity with respect to β was studied. The precessional factor β is generally considered to be a constant for physical reasons [3]. However in [2] we allowed it to be a space-dependent function to describe the geometry of the ferromagnetic core in magnetoresistive random access memories (MRAM). We studied the optimal design of the MRAM core. Other possible applications were mentioned in [4] including the determination of a non-constant anisotropy parameter for grained media and the shape optimization of the working domain. For composite materials, the damping constant α can also be a space-dependent function. Different components of the material may have various damping constants. This inhomogeneity is difficult to measure directly and must be determined indirectly from measurements of the magnetic field.

Our workflow follows this simple scheme.

Definition of the cost functional. Given the measurements we are able to compare the current approximation of the magnetization with these measurements. This can be done by measuring the distance between $\mathbf{m}^*(t)$ and $\mathbf{m}(\alpha, \beta)$. We choose this distance to be an integral over a time interval from the L^2 difference.

Minimization of the cost functional. Next step is to choose a method for the minimization. We employ the steepest descend method. Gradient-based iterative methods are preferable if first-order information is available. They converge significantly faster than zero-order methods and orders of magnitude faster than genetic algorithms. However, in general the global convergence cannot be justified. Genetic algorithms usually find better solutions.

Computation of the gradients. For the gradient-based methods one needs to evaluate derivatives of the cost functional with respect to the parameters α, β. We employ the adjoint variable method for this purpose, which becomes very effective for large dimensions of the parameter space.

Convergence of the minimization algorithm. Having the gradients evaluated, we can generate a sequence of approximations α_n, β_n by the steepest descent method.[1] We need to discuss the possible convergence of this sequence to a minimizer of the cost functional.

Numerical examples. We provide several computational studies illustrating the usefulness of the proposed algorithm.

The paper is organized as follows. The direct problem is presented in detail in Section 2. In Section 3 we describe the inverse problem. We construct a cost functional that measures how good current approximations of α and β are. We define directional derivatives of the cost functional and set up the corresponding sensitivity equations.

In Section 4 we focus on practical aspects of the computation of the gradients. We remark that the knowledge of a PDE, solution of which is the directional derivative of the cost functional, is insufficient for practical implementations. It leads to huge computational effort needed to be done. We suggest an alternative approach using the adjoint variable method. The proposed method uses a solution to an adjoint problem that rapidly speeds up the computations.

We discuss the convergence of the steepest descent method in Section 5. We verify a relaxation result in Theorem 2. Due to the high nonlinearity of the underlying direct problem we were not able to get convergence to the minimizer.

In Section 6, we provide the numerical implementation of the problem. We present several examples of the determination of both α and β.

The theorems concerning uniqueness, existence, regularity and stability of the solutions to the direct problem and to the sensitivity equations are all moved to Appendices A–C to increase the readability of the text. The sensitivity analysis performed in the Appendixes is a continuation of the work done in [2]. We point out that these results are new and they are necessary to establish the main result, i.e., that the steepest descent method generates a relaxation sequence.

Notations

By Ω we denote a bounded domain representing a magnetic workpiece and by $\partial\Omega$ its boundary. The evolution time interval is denoted by $I := (0, T)$. We use symbol $\langle \cdot, \cdot \rangle$ for the scalar product of two vectors in $\mathbb{R}^d$ space, d being the dimension. By $(\cdot, \cdot)$ we denote the standard scalar product in $L^2(\Omega)$ space and by $\| \cdot \|_p$ we understand the norm in $L^p(\Omega)$ space, $1 \le p \le \infty$. By ∂_ξ we denote partial derivative with respect to ξ. By $[\partial_\xi \cdot, \cdot]$ we denote formal duality between the partial directional derivative of a function and the direction in which the derivative is considered. In the whole text, the magnetization $\mathbf{m} = \mathbf{m}(\mathbf{x}, t)$ is always considered as a function of time and space. For simplicity we omit $\mathbf{x}$ and t. The same implies to solutions to the sensitivity equations and to the dual equations.

2. Direct problem

Let us rigorously define the direct problem. By scalar multiplication of the LL equation (1) we directly see that the modulus of $\mathbf{m}$ remains unchanged throughout the time evolution. This is valid only for the temperatures below the Curie temperature. The thermal effects appearing around this threshold have been recently studied in [5]. On account of the conservation of the modulus we can study the LL equation in an equivalent form [6].

---

[1] Other gradient-based methods e.g. conjugate gradients (CG) can be used.

Problem 1. For given $\alpha, \beta \in L^\infty(\Omega) \cap W^{1,2}(\Omega)$ find $\mathbf{m} \in L^\infty((0,T), W^{1,2}(\Omega)) \cap L^2((0,T), W^{2,2}(\Omega))$ such that $\partial_t \mathbf{m} \in L^2((0,T), L^2(\Omega))$, satisfying the LL equation

$$\begin{aligned} &\partial_t \mathbf{m} - \alpha \Delta \mathbf{m} = \alpha |\nabla \mathbf{m}|^2 \mathbf{m} - \beta \mathbf{m} \times \Delta \mathbf{m}, \quad \text{in } (0,T) \times \Omega, \\ &\langle \nabla \mathbf{m}, \mathbf{n} \rangle = 0, \quad \text{on } (0,T) \times \partial\Omega, \qquad \mathbf{m}(0,\mathbf{x}) = \mathbf{m}^0(\mathbf{x}) \quad \text{in } \Omega. \end{aligned} \tag{3}$$

For the direct Problem 1, from [7] we have the existence and the uniqueness, assuming that the initial condition have small $W^{1,2}(\Omega)$ norm and belong to $W^{2,2}(\Omega)$. This result can be extended for arbitrary regularity assuming smooth initial condition using the program elaborated in [6, Theorem 4.3].

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded regular domain. Assume that $\mathbf{m}_0 \in W^{k,2}(\Omega)$. There exists a constant δ such that if $\|\nabla \mathbf{m}_0\|_2 \le \delta$, then there exists a unique solution $\mathbf{m}$ to Problem 1 satisfying*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{m}(t)\|_{W^{k,2}} + \left[\int_0^T \|\mathbf{m}\|^2_{W^{k+1,2}}\right]^{\frac{1}{2}} \le C.$$

We can use the boundedness of $\mathbf{m}$ in $W^{2,2}(\Omega)$ and thus from the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ we have also the boundedness of $\mathbf{m}$ in the spaces $W^{1,4}(\Omega)$ and $L^\infty(\Omega)$.

3. Inverse problem

Computing the magnetization $\mathbf{m}(\alpha, \beta)$ as the solution to Problem 1 for any α, β defines the direct problem. The inverse problem we consider consists of determining the optimal functions $\alpha_{\text{opt}}, \beta_{\text{opt}}$ as arg $\min_{\alpha,\beta\in\mathcal{Q}} F(\alpha, \beta)$ for some cost functional F involving measured data $\mathbf{m}^*$. We employ the following cost functional

$$F(\alpha, \beta) = \frac{1}{2} \int_0^T \|\mathbf{m}(\alpha, \beta) - \mathbf{m}^*\|_2^2. \tag{4}$$

Note that F can be chosen differently, according to the needs of the specific application. We study the determination of α and β separately. We always consider one of these functions fixed and the second being optimized.

We aim at minimizing the cost functional by an iterative procedure generating α_n or β_n. We choose the steepest descent method so our sequence will be generated according to

$$\alpha_{n+1} = \alpha_n - \lambda_n \partial_\alpha F(\alpha_n, \beta), \tag{5}$$

with an analogous sequence for β_n. The evaluation of the directional derivatives $\partial_\alpha F$, $\partial_\beta F$ is crucial. Let us note that once they are known, methods converging faster than the steepest descent method can be used, for example the conjugate gradient method. We now focus on how the directional derivatives can be computed. We will work with Gâteaux derivatives in a specific direction represented by a function in $\mathcal{Q}$. The formal differentiation of $F(\alpha, \beta)$ with respect to α in direction μ_1 and with respect to β in direction μ_2 gives

$$[\partial_\alpha F, \mu_1] = \int_0^T (\mathbf{v}_1, \mathbf{m} - \mathbf{m}^*), \qquad [\partial_\beta F, \mu_2] = \int_0^T (\mathbf{v}_2, \mathbf{m} - \mathbf{m}^*), \tag{6}$$

where we used the following notations

$$\mathbf{v}_1 := [\partial_\alpha \mathbf{m}, \mu_1] := \lim_{\varepsilon\to 0} \frac{\mathbf{m}(\alpha + \varepsilon\mu_1, \beta) - \mathbf{m}(\alpha, \beta)}{\varepsilon}, \qquad \mathbf{v}_2 := \left[\partial_\beta \mathbf{m}, \mu_2\right] := \lim_{\varepsilon\to 0} \frac{\mathbf{m}(\alpha, \beta + \varepsilon\mu_2) - \mathbf{m}(\alpha, \beta)}{\varepsilon}.$$

Functions μ_1, μ_2 belong to the same function spaces as α and β do, respectively. The meaning of $\mathbf{v}_1, \mathbf{v}_2$ is intuitive. The $L^\infty(\Omega)$ norm of the nominators in the limit is bounded. This follows from the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ and from Theorems 4 and 5 for $\mathbf{v}_1, \mathbf{v}_2$, respectively. This means that $\mathbf{v}_1, \mathbf{v}_2$ actually exist and each satisfies the corresponding sensitivity equation. For $\mathbf{v}_1$ we formally differentiate the LL equation with respect to α and for $\mathbf{v}_2$ with respect to β obtaining the sensitivity equations

$$\begin{aligned} \partial_t \mathbf{v}_1 - \alpha \Delta \mathbf{v}_1 &= \alpha R(\mathbf{m}, \mathbf{v}_1) + \beta S(\mathbf{m}, \mathbf{v}_1) + \mu_1 P_1(\mathbf{m}), \\ \partial_t \mathbf{v}_2 - \alpha \Delta \mathbf{v}_2 &= \alpha R(\mathbf{m}, \mathbf{v}_2) + \beta S(\mathbf{m}, \mathbf{v}_2) + \mu_2 P_2(\mathbf{m}), \end{aligned} \tag{7}$$

where we used the following notations

$$\begin{aligned} &P_1(\mathbf{m}) := \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m}, && P_2(\mathbf{m}) := -\mathbf{m} \times \Delta \mathbf{m}, \\ &R(\mathbf{m}, \mathbf{u}) := 2\langle \nabla \mathbf{u}, \nabla \mathbf{m} \rangle \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{u}, && S(\mathbf{m}, \mathbf{u}) := -\mathbf{u} \times \Delta \mathbf{m} - \mathbf{m} \times \Delta \mathbf{u}. \end{aligned}$$

Theory for the direct problem is well established. However for sensitivity equations, to the best knowledge of the authors, no existence, uniqueness or regularity results are available in the literature. We establish such theoretical results in Theorem 3.

Another important theoretical result concerns the stability of the direct problem and of sensitivity equations on parameters α and β. We show in Appendices B and C that the solutions to the direct problem and to the sensitivity equations are stable with respect to the perturbations in α and β. These results are summarized in Theorems 4–7. They play a key role in the proof of the Lipschitz continuity of $\partial_\alpha F$ and of $\partial_\beta F$ in Section 5.

4. Dual problem and its motivation

Now we focus on the actual computation of the Gâteaux derivatives of $\mathbf{m}$. We use Finite Element Method to solve our problem. Suppose that a two dimensional rectangular domain $\Omega = (0,1)^2$ is discretized by a regular triangular mesh with diameter 1/50. Using the Lagrange elements of the first order with degree of freedom linked to the vertex of a mesh we have $d = 51^2$ elements. That means that our parameter space has the dimension $d = 2601$ and the cost functional $F(\alpha, \beta)$ depends on $2d$ unknowns. For the optimization algorithm (5) we need to evaluate the partial derivative of F with respect to α. This leads to the evaluation of $[\partial_\alpha F, \mu_1]$ for d different vectors μ_1. At this stage, if μ_1 is given we are able to compute $[\partial_\alpha F, \mu_1]$ by solving one PDE. Thus the evaluation of $\partial_\alpha F$ requires in total d solutions of some PDE. Obviously, this is not practically possible.

To solve this problem we use the adjoint variable method. We introduce the variable $\boldsymbol{\varphi}$, the solution of the adjoint problem which we specify below.

Consider the weak formulation of the sensitivity equation. By $A(\mathbf{v}, \boldsymbol{\varphi})$ denote the operator part of the equation

$$A(\mathbf{v}, \boldsymbol{\varphi}) := (\partial_t \mathbf{v}, \boldsymbol{\varphi}) - (\alpha \Delta \mathbf{v}, \boldsymbol{\varphi}) - (\alpha R(\mathbf{m}, \mathbf{v}), \boldsymbol{\varphi}) - (\beta S(\mathbf{m}, \mathbf{v}), \boldsymbol{\varphi}).$$

Now we move all the time and the space derivatives of $\mathbf{v}$ to the adjoint variable $\boldsymbol{\varphi}$. Using the matrix identity $(\mathbf{a} \times \mathbf{b}, \mathbf{c}) = (\mathbf{b} \times \mathbf{c}, \mathbf{a})$ for some terms we get

$$\begin{aligned}(\alpha R(\mathbf{m}, \mathbf{v}) + \beta S(\mathbf{m}, \mathbf{v}), \boldsymbol{\varphi}) &= 2(\alpha \langle \nabla \mathbf{v}, \nabla \mathbf{m} \rangle \mathbf{m} + \alpha |\nabla \mathbf{m}|^2 \mathbf{v}, \boldsymbol{\varphi}) - (\beta \mathbf{v} \times \Delta \mathbf{m} - \beta \mathbf{m} \times \Delta \mathbf{v}, \boldsymbol{\varphi}) \\ &= -2(\nabla \cdot (\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \nabla \mathbf{m}), \mathbf{v}) - 2(\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \langle \nabla \mathbf{m}, \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega} \\ &\quad + (\alpha |\nabla \mathbf{m}|^2 \boldsymbol{\varphi}, \mathbf{v}) - (\beta \Delta \mathbf{m} \times \boldsymbol{\varphi}, \mathbf{v}) - (\beta \boldsymbol{\varphi} \times \mathbf{m}, \Delta \mathbf{v}).\end{aligned}$$

In the second term on the r.h.s. we use the boundary condition $\langle \nabla \mathbf{m}, \mathbf{n} \rangle = 0$ on $(0, T) \times \partial\Omega$. We proceed with the last term on the r.h.s. Solutions to the sensitivity equations take over the initial conditions from Problem 1 so we have $\langle \nabla \mathbf{v}, \mathbf{n} \rangle = 0$ on $(0, T) \times \partial\Omega$ and subsequently we get

$$(\beta \boldsymbol{\varphi} \times \mathbf{m}, \Delta \mathbf{v}) = -(\nabla(\beta \boldsymbol{\varphi} \times \mathbf{m}), \nabla \mathbf{v}) = (\Delta(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{v}) - (\langle \nabla(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega}.$$

Consequently we get

$$\begin{aligned}(\alpha R(\mathbf{m}, \mathbf{v}) + \beta S(\mathbf{m}, \mathbf{v}), \boldsymbol{\varphi}) &= -2(\nabla \cdot (\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \nabla \mathbf{m}), \mathbf{v}) + (\alpha |\nabla \mathbf{m}|^2 \boldsymbol{\varphi}, \mathbf{v}) - (\beta \Delta \mathbf{m} \times \boldsymbol{\varphi}, \mathbf{v}) \\ &\quad - (\Delta(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{v}) + (\langle \nabla(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega}.\end{aligned} \tag{8}$$

Realize that $\mathbf{v}(0, \mathbf{x}) = \mathbf{0}$. We put $\boldsymbol{\varphi}(T, \mathbf{x}) = \mathbf{0}$. Thus we get

$$\begin{aligned}(\partial_t \mathbf{v}, \boldsymbol{\varphi}) - (\alpha \Delta \mathbf{v}, \boldsymbol{\varphi}) &= -(\mathbf{v}, \partial_t \boldsymbol{\varphi}) + (\nabla(\alpha \boldsymbol{\varphi}), \nabla \mathbf{v}) - (\langle \nabla \mathbf{v}, \mathbf{n} \rangle, \alpha \boldsymbol{\varphi})_{\partial\Omega} \\ &= -(\mathbf{v}, \partial_t \boldsymbol{\varphi}) - (\Delta(\alpha \boldsymbol{\varphi}), \mathbf{v}) + (\langle \nabla(\alpha \boldsymbol{\varphi}), \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega},\end{aligned} \tag{9}$$

where we have used $\langle \nabla \mathbf{v}, \mathbf{n} \rangle = 0$ on $(0, T) \times \partial\Omega$ again. Summarizing (8) and (9) we get

$$\begin{aligned}A(\mathbf{v}, \boldsymbol{\varphi}) = &-(\mathbf{v}, \partial_t \boldsymbol{\varphi}) - (\Delta(\alpha \boldsymbol{\varphi}), \mathbf{v}) + 2(\nabla \cdot (\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \nabla \mathbf{m}), \mathbf{v}) - (\alpha |\nabla \mathbf{m}|^2 \boldsymbol{\varphi}, \mathbf{v}) \\ &+ (\beta \Delta \mathbf{m} \times \boldsymbol{\varphi}, \mathbf{v}) + (\Delta(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{v}) \\ &+ (\langle \nabla(\alpha \boldsymbol{\varphi}), \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega} - (\langle \nabla(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{n} \rangle, \mathbf{v})_{\partial\Omega}.\end{aligned}$$

We prescribe

$$\langle \nabla(-\alpha \boldsymbol{\varphi} + \beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{n} \rangle = 0 \quad \text{on } (0, T) \times \partial\Omega \tag{10}$$

and thus we get rid of the boundary terms obtaining

$$\begin{aligned}A(\mathbf{v}, \boldsymbol{\varphi}) = &-(\mathbf{v}, \partial_t \boldsymbol{\varphi}) - (\Delta(\alpha \boldsymbol{\varphi}), \mathbf{v}) + 2(\nabla \cdot (\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \nabla \mathbf{m}), \mathbf{v}) - (\alpha |\nabla \mathbf{m}|^2 \boldsymbol{\varphi}, \mathbf{v}) \\ &+ (\beta \Delta \mathbf{m} \times \boldsymbol{\varphi}, \mathbf{v}) + (\Delta(\beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{v}).\end{aligned} \tag{11}$$

We now define the adjoint problem.

Problem 2. Find $\boldsymbol{\varphi} \in L^\infty((0, T), W^{2,2}(\Omega))$ such that

$$-\partial_t \boldsymbol{\varphi} - \Delta(\alpha \boldsymbol{\varphi}) + 2\nabla \cdot (\alpha \langle \mathbf{m}, \boldsymbol{\varphi} \rangle \nabla \mathbf{m}) - \alpha |\nabla \mathbf{m}|^2 \boldsymbol{\varphi} + \beta \Delta \mathbf{m} \times \boldsymbol{\varphi} + \Delta(\beta \boldsymbol{\varphi} \times \mathbf{m}) = (\mathbf{m} - \mathbf{m}^*) \quad \text{in } (0, T) \times \Omega, \tag{12}$$

$$\boldsymbol{\varphi}(T, \mathbf{x}) = \mathbf{0} \quad \text{in } \Omega \tag{13}$$

$$\langle \nabla(-\alpha \boldsymbol{\varphi} + \beta \boldsymbol{\varphi} \times \mathbf{m}), \mathbf{n} \rangle = 0 \quad \text{on } (0, T) \times \partial\Omega. \tag{14}$$

Eq. (12) is an adjoint equation of the sensitivity Eq. (7). Consequently the theoretical results obtained in Appendix A can be easily adapted to show that Problem 2 has a unique solution.

We can multiply (12) by $\mathbf{v}$. Using (11) and subsequently (7) we get

$$(\mathbf{v}, \mathbf{m} - \mathbf{m}^*) = A(\mathbf{v}, \boldsymbol{\varphi}) = (\mu_i P_i(\mathbf{m}), \boldsymbol{\varphi}),$$

and as a conclusion we can state the following lemma.

Lemma 1. *Gâteaux derivatives of*

$$F(\alpha, \beta) := \frac{1}{2} \int_0^T \|\mathbf{m}(\alpha, \beta) - \mathbf{m}^*\|_2^2 \mathrm{d}t \tag{15}$$

with respect to α, β in directions μ_1, μ_2 can be expressed as

$$[\partial_\alpha F, \mu_1] = \int_0^T (\mathbf{v}_1, \mathbf{m} - \mathbf{m}^*)\mathrm{d}t = \int_0^T (\mu_1 P_1(\mathbf{m}), \boldsymbol{\varphi})\mathrm{d}t,$$
$$[\partial_\beta F, \mu_2] = \int_0^T (\mathbf{v}_2, \mathbf{m} - \mathbf{m}^*)\mathrm{d}t = \int_0^T (\mu_2 P_2(\mathbf{m}), \boldsymbol{\varphi})\mathrm{d}t,$$

where $\boldsymbol{\varphi}$ is the solution to the adjoint Problem 2.

The main advantage of this lemma is that we solve only one PDE obtaining $\boldsymbol{\varphi}$ as a solution to the adjoint problem. Then, to evaluate the gradient of $\partial_\alpha F$, for each $\mu_1 = \mathbf{e}_i$ we do not have to find corresponding $\mathbf{v}_1$, it is sufficient to recompute the scalar product in $L^2(I, L^2(\Omega))$. Only d scalar products and one PDE is necessary to obtain the gradient of F.

5. Relaxation process

The purpose of this section is to prove that (5) is a relaxation sequence. From the rather general result [8, Lemma 11.2] we have the convergence of (5) to a stationary point of F. We rewrite the cited lemma.

Lemma 2. *Suppose that $\mathcal{Q}$ is a real reflexive Banach space and $\mathcal{Q}^*$ is a space with Gâteaux-differentiable norm. Further assume $F(x)$ is a real Gâteaux-differentiable functional bounded below and increasing, and its gradient satisfies the Lipschitz condition $\|DF(x+h) - DF(x)\|_{\mathcal{Q}^*} \le C_F \|h\|_{\mathcal{Q}}$.*

Then $x_{n+1} = x_n - \lambda_n DF(x_n)$ is a relaxation process and $\lim_{n\to\infty} DF(x_n) = 0$ as soon as $1/4 \le \lambda_n C_F \le 1/2$.

We verify if our setting satisfies the hypotheses of the lemma. The assumption that $\mathcal{Q}$ is a reflexive Banach space can be relaxed. It suffices that $\mathcal{Q}$ is the dual of a normed separable space, see the note in the beginning of Chapter III of [8]. In our case $\mathcal{Q} = L^\infty(\Omega) = (L^1(\Omega))^*$.

The second statement requires the functional $F(x)$ to be increasing to ensure that the relaxation sequence is bounded. A sufficient condition for $F(x)$ to be increasing is to satisfy

$$\lim_{\|x\|_{\mathcal{Q}} \to \infty} F(x) = +\infty. \tag{16}$$

To guarantee this, we add to the functional (4) a regularization term

$$\eta_\alpha \|\alpha\|_{\mathcal{Q}}^2 + \eta_\beta \|\beta\|_{\mathcal{Q}}^2, \tag{17}$$

where $\eta_\alpha, \eta_\beta > 0$ are small regularization constants. Note that in $\partial_\alpha(F)$, the β regularization vanishes and similarly in $\partial_\beta(F)$, the α regularization vanishes. Further, DF must be Lipschitz continuous which will be verified in the following lemma.

Lemma 3. *Gâteaux derivatives $\partial_\alpha F$ and $\partial_\beta F$ are Lipschitz continuous with respect to $\mathcal{Q} = L^\infty(\Omega)$ space.*

Proof. We begin with $\partial_\alpha F$. Using Lemma 1 we estimate

$$\begin{aligned}
&\|\partial_\alpha F(\alpha+h, \beta) - \partial_\alpha F(\alpha, \beta)\|_{\mathcal{Q}^*} = \sup_{\|\mu_1\|_{\mathcal{Q}}=1} |[\partial_\alpha F(\alpha+h, \beta) - \partial_\alpha F(\alpha, \beta), \mu_1]| \\
&\quad = \sup_{\|\mu_1\|_{\mathcal{Q}}=1} \left| \int_0^T (\mathbf{v}_1(\alpha+h, \beta), \mathbf{m}(\alpha+h, \beta) - \mathbf{m}^*)\mathrm{d}t - \int_0^T (\mathbf{v}_1(\alpha, \beta), \mathbf{m}(\alpha, \beta) - \mathbf{m}^*)\mathrm{d}t \right| \\
&\quad \le C \sup_{\|\mu_1\|_{\mathcal{Q}}=1} \int_0^T \Big[\|\mathbf{v}_1(\alpha+h, \beta) - \mathbf{v}_1(\alpha, \beta)\|_2 \|\mathbf{m}(\alpha+h, \beta) - \mathbf{m}^*\|_\infty \\
&\qquad + \|\mathbf{v}_1(\alpha, \beta)\|_\infty \|\mathbf{m}(\alpha+h, \beta) - \mathbf{m}(\alpha, \beta)\|_2 \Big] \mathrm{d}t.
\end{aligned}$$

For the four norms we subsequently use Theorems 1, 3, 4 and 6 to get the Lipschitz continuity

$$\|\partial_\alpha F(\alpha + h, \beta) - \partial_\alpha F(\alpha, \beta)\|_{\mathcal{Q}^*} \le C\|h\|_{\mathcal{Q}}. \tag{18}$$

For $\partial_\beta F$ we can do the same. At the end of the proof we use Theorems 1, 3, 5 and 7 to get the Lipschitz continuity of $\partial_\beta F$. $\square$

The last condition from Lemma 2 is that $1/4 \le \lambda_n C_F \le 1/2$. This can be fulfilled by the actual implementation.

After verification of all conditions of Lemma 2, we state the following theorem concerning the convergence of our minimizing sequences.

Theorem 2. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. Assume that $1/4 \le \lambda_n C_F \le 1/2$ and $1/4 \le \delta_n C_F \le 1/2$ where C_F is the Lipschitz constant from Lemma 3. Then the following iterative processes*

$$\alpha_{n+1} = \alpha_n - \lambda_n \partial_\alpha F(\alpha_n, \beta), \qquad \beta_{n+1} = \beta_n - \delta_n \partial_\beta F(\alpha, \beta_n)$$

are relaxations, i.e. $\lim_{n\to\infty} \partial_\alpha F(\alpha_n, \beta) = 0$ *and* $\lim_{n\to\infty} \partial_\beta F(\alpha, \beta_n) = 0$.

6. Numerical implementation

Algorithm. We use Algorithm 1 based on the steepest descend method to minimize the functional F defined by (4). For clarity, we present the algorithm for the optimization of the damping parameter function α only (β being fixed). In the case of β the structure of the algorithm is identical. Notice that each iteration of the algorithm consists of three major parts: the computation of the steepest direction, the determination of the optimal step and the update.

The first part is the one where the use of the adjoint variable method means a significant reduction of the computational time. For adjoint variable method we need to solve only two partial differential equations in comparison with $d + 1$ PDEs when the sensitivity equation is used directly. Here d is the dimension of the approximation space.

Data: $n = 0$; $\alpha_n = \alpha_0$;
do
  Compute $\partial_\alpha F(\alpha_n)$:
    $\alpha_n \longrightarrow$ direct Problem 1 $\longrightarrow \mathbf{m}_n$
    $(\alpha_n, \mathbf{m}_n) \longrightarrow$ adjoint Problem 2 $\longrightarrow \varphi_n$
    $(\alpha_n, \mathbf{m}_n, \varphi_n) \longrightarrow$ Lemma 1 $\longrightarrow \partial_\alpha F(\alpha_n)$
  Determine the optimal step λ_n
    linesearch$(\alpha_n, F(\alpha_n), \partial_\alpha F(\alpha_n)) \longrightarrow$ optimal step λ_n
  Update the current approximation α_n
    $\alpha_{n+1} = \alpha_n - \lambda_n \partial_\alpha F(\alpha_n))$
    $n{+}{+}$
while $[F(\alpha_n) - F(\alpha_{n+1})]/F(\alpha_n) > \epsilon$

Algorithm 1: Implementation of the steepest descend method.

The second part is common for iterative methods. It can e.g. involve the linesearch algorithm for determination of the optimal step length. This might be quite time-consuming, since the linesearch detects the optimal λ_n by the evaluation of the cost functional for different intermediate values of λ_n and one such evaluation means to solve one direct Problem 1. However, we do not need to find the optimal value of λ_n for which the drop of F is maximal. It is enough to find one value for which F drops sufficiently (the method is then no more steepest descent). We update λ according to the following simple rule

$$\lambda_n = 2\lambda_{n-1} \quad \text{if } F(\alpha_n(\lambda_{n-2})) < F(\alpha_{n-1}),$$

i.e. when $\lambda_{n-1} := \lambda_{n-2}$ gave a reduction of cost functional value, we try double the step. If in the next step λ_n does not give a descent, we take the lambda with the smallest k from the sequence $\lambda_n^k = \lambda_n^{k-1}/2, k = 1, \ldots, \infty$ such that we have a descent.

The last part is the actual update process. Algorithm 1 stops when the relative speed of reduction of functional values drops under certain threshold ϵ. In the computations below we take $\epsilon = 0.001$. This stopping criterion is very easy but quite effective even in the case of a noisy data and for the cost functional without an added regularization term.[2] Other choices of stopping criteria are possible, e.g by the discrepancy principle [9] the algorithm stops when residuum drops under an expected level of noise. No regularization term is added to the cost functional. In fact the number of iterations plays the role of regularization parameter. Such an algorithm converges for exact data under certain conditions to the least square solution of minimal L^2 norm. Other possibility is to use the norm of the gradient as a stopping rule. This has to be combined

[2] We needed such a term to ensure that the cost functional is increasing in Section 5.

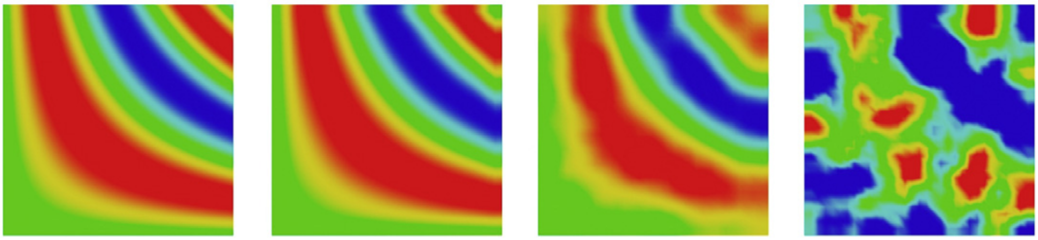

Fig. 1. Determination of the smooth damping parameter. The exact solution and the reconstructions with different noise levels of 0%, 1% and 10%.

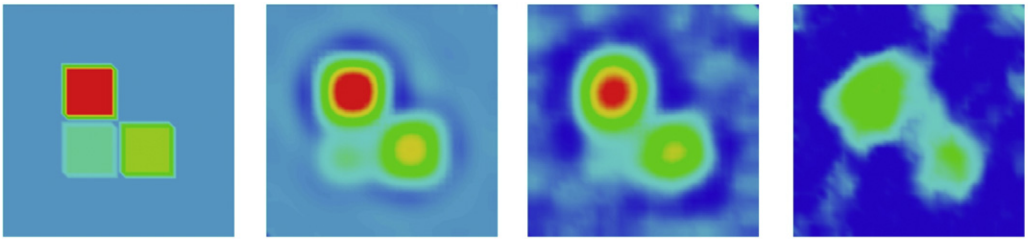

Fig. 2. Determination of piecewise constant damping parameter. The exact solution and the reconstructions with different noise levels of 0%, 1% and 10%.

with addition of some regularization term to the cost functional in the noisy case. Different choices are possible, e.g. total variation regularization which preserves sharp edges. For regularization of inverse problem see e.g. [10].

Approximation. To solve the direct Problem 1, we employ the method developed in [11], where the convergence and the error estimates of time discretization are proved. This method is based on the variational formulation of (3), so the standard $W^{1,2}(\Omega)$-conforming finite elements are used to discretize in space.

The magnetization preserves length $|\mathbf{m}| = 1$. The above mentioned method preserves the length of the magnetization only asymptotically. Therefore we project the magnetization to keep the constant length and to stabilize the method.

As explained in Section 4, the adjoint problem is a linear parabolic PDE. Thus we also use the standard $W^{1,2}(\Omega)$-conforming finite element formulation. The simple implicit Backward Euler approximation of the time derivative is used.

For numerical realization, we used FreeFem++ package [12].

Test with smooth exact solution α. The first numerical test was run with the following continuous exact solution

$$\alpha_{exact} = 0.02 + 0.01 \sin(\pi^2 xy), \quad (x, y) \in (0, 1)^2.$$

The precession factor β was fixed to 0.1. All the finite element functions were defined on a regular mesh with diameter $h = 1/30$, i.e. the dimension of the approximation space was $d = 961$. The time step of numerical schemes was $\tau = 0.01$ and $T = 15\tau$. The reconstruction of the exact profile depicted in Fig. 1 for measurements without noise was almost perfect. When we added noise of the level 1%, the reconstruction became worse but still acceptable. Increasing the noise to 10% resulted in the loss of almost any information.

Test with discontinuous exact solution α. Next numerical test was run with a discontinuous piecewise constant solution having 4 different values 0.005, 0.01, 0.02 and 0.03 in $\Omega = (0, 1)^2$. The other parameters were again $h = 1/30$, $\tau = 0.01$ and $T = 15\tau$. The profile is depicted in Fig. 2. We see that already the reconstruction of the exact profile was not so successful as for the case of the smooth exact solution. The explanation is in the regularization that has been used. The L^2-regularization [10] introduced by algorithm causes a smoothing effect that is harmless in the case of the exact solution but destroys the information about sharp edges in the case of the discontinuous exact solution. When we added noise of the level 1%, we see that the reconstruction became almost unacceptable.

Test with smooth exact solution β. Reconstruction of β delivered very similar results to those for α. We present only the case of a smooth exact solution

$$\beta_{exact} = 0.2 + 0.1 \sin[2\pi(x + y)], \quad (x, y) \in (0, 1)^2.$$

We fixed $\alpha = 0.01$. All the other settings were the same as in the numerical examples for α. The results of reconstruction for different noise levels are presented in Fig. 3. In Fig. 4 one can find an iteration process for the noise level of 1%. The algorithm needed only nine iterations to converge.

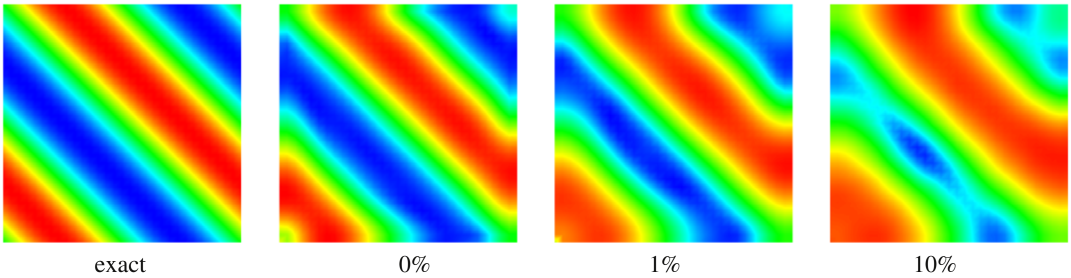

Fig. 3. Determination of smooth precession parameter. The exact solution and the reconstructions with different noise levels of 0%, 1% and 10%.

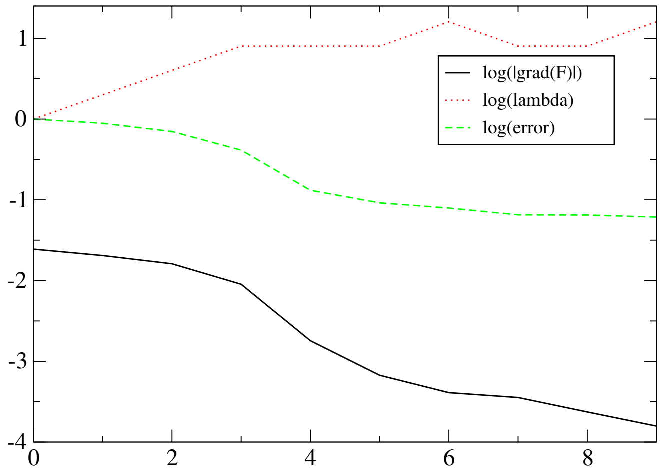

Fig. 4. Convergence for noise level of 1%; iteration number n versus the norm of gradient $\|\partial_\beta F(\beta_n)\|$, the step length λ_n and the relative error between β_n and β_{exact}.

Acknowledgements

The first author is supported by the Fund for Scientific Research – Flanders FWO (Belgium). The second author is supported by the GOA project BOF 07/GOA/006 of Ghent University.

Appendix A. Existence, uniqueness and regularity results for the sensitivity equations

In this theoretical section we will frequently make use of the following generalized Sobolev inequality in two dimensions

$$\|u\|_4^2 \le C\|u\|_2\|\nabla u\|_2 \tag{19}$$

which remains valid for the vector valued functions $\mathbf{u}$. Further we use the weighted Young inequality

$$a.b \le \epsilon^p \frac{a^p}{p} + \frac{1}{\epsilon^q}\frac{b^q}{q} \tag{20}$$

which is valid for a positive ϵ and positive $p, q > 1$ such that $p^{-1} + q^{-1} = 1$. Since ϵ can be chosen arbitrarily small, we can consider the previous inequality in the form $a.b \le \varepsilon a^p + C_\varepsilon b^q$, where ε is a small and C_ε a large positive parameter.

We show the existence and uniqueness result for the sensitivity equations. The sensitivity equations (7) are linear parabolic systems with a principal operator

$$L\mathbf{u} := -\alpha \Delta \mathbf{u} - \alpha R(\mathbf{m}, \mathbf{u}) - \beta S(\mathbf{m}, \mathbf{u}).$$

$L\mathbf{u}$ is a second-order partial differential operator whose coefficients are functions of $\Delta \mathbf{m}$, $\nabla \mathbf{m}$, $\mathbf{m}$. Using Theorem 1 for $k = 4$ and the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, we can ensure the boundedness of these functions in $W^{2,2}(\Omega)$ and subsequently also in $L^\infty(\Omega)$. Thus the coefficients of the operator in L are $L^\infty(\Omega)$ bounded. Using these results we verify the conditions of the well-established theory for such kind of parabolic systems. From [13, Theorem 5, Section 7.1] we can conclude the following result.

Theorem 3. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. Then there exist a unique solution $\mathbf{v}_i$ to the sensitivity equation (7) for $i = 1, 2$ respectively satisfying*

$$\operatorname*{ess\,sup}_{t\in I}(\|\mathbf{v}_i\|_{W^{2,2}} + \|\partial_t \mathbf{v}_i\|_2) + \left[\int_0^T \|\partial_t \mathbf{v}_i\|^2_{W^{1,2}}\right]^{\frac{1}{2}} \le C\|\mu_i\|_\infty. \tag{21}$$

Appendix B. Stability of the direct problem

The α-stability. Take two functions α^1 and α^2 and suppose $\mathbf{m}^i$ are solutions to Problem 1 for function α^i, $i = 1, 2$ with the same initial condition. Then the difference $\overline{\mathbf{m}} := \mathbf{m}^1 - \mathbf{m}^2$ obeys the following equation

$$\begin{aligned}\partial_t \overline{\mathbf{m}} - \alpha^2 \Delta \overline{\mathbf{m}} &= (\alpha^1 - \alpha^2) P_1(\mathbf{m}^1) + \alpha^2\Big[\langle \nabla \overline{\mathbf{m}}, \nabla \mathbf{m}^1 + \nabla \mathbf{m}^2\rangle \mathbf{m}^1 + |\nabla \mathbf{m}^2|^2 \overline{\mathbf{m}}\Big] \\ &\quad - \beta\Big[\overline{\mathbf{m}} \times \Delta \mathbf{m}^1 + \mathbf{m}^2 \times \Delta \overline{\mathbf{m}}\Big],\end{aligned} \tag{22}$$

with zero initial condition. This equation has basically the same structure as the sensitivity equation; the principal operator of (22) is slightly different. Nevertheless we can again verify the assumptions for [13, Theorem 5, Section 7.1] and thus we can derive an analogical result stated in the following theorem.

Theorem 4. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. For an arbitrary $\alpha^i \in L^\infty(\Omega)$, $i = 1, 2$ consider Problem 1_i derived from Problem 1 simply by replacing α with α^i and denote by $\mathbf{m}^i$ the solution to Problem 1_i. Then the difference $\overline{\mathbf{m}} := \mathbf{m}^1 - \mathbf{m}^2$ enjoys*

$$\operatorname*{ess\,sup}_{t\in I}(\|\overline{\mathbf{m}}\|_{W^{2,2}} + \|\partial_t \overline{\mathbf{m}}\|_2) + \left[\int_0^T \|\partial_t \overline{\mathbf{m}}\|^2_{W^{1,2}}\right]^{\frac{1}{2}} \le C(\|\alpha^1 - \alpha^2\|_\infty).$$

The β-stability. Similarly as before, take two functions β^1 and β^2 and suppose $\mathbf{m}^i$ are solutions to Problem 1 for function β^i, $i = 1, 2$ with the same initial condition. Then for the difference $\overline{\mathbf{m}} = \mathbf{m}^1 - \mathbf{m}^2$ we can write

$$\begin{aligned}\partial_t \overline{\mathbf{m}} - \alpha \Delta \overline{\mathbf{m}} &= (\beta^1 - \beta^2) P_2(\mathbf{m}^1) + \alpha\Big[\langle \nabla \overline{\mathbf{m}}, \nabla \mathbf{m}^1 + \nabla \mathbf{m}^2\rangle \mathbf{m}^1 + |\nabla \mathbf{m}^2|^2 \overline{\mathbf{m}}\Big] \\ &\quad - \beta^2\Big[\overline{\mathbf{m}} \times \Delta \mathbf{m}^1 + \mathbf{m}^2 \times \Delta \overline{\mathbf{m}}\Big],\end{aligned} \tag{23}$$

with the zero initial condition. Again, the equation has the same structure as the sensitivity equation. Thus an analogical result is stated in the following theorem.

Theorem 5. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. For an arbitrary $\beta^i \in L^\infty(\Omega)$, $i = 1, 2$ consider Problem 1_i derived from Problem 1 simply by replacing β with β^i and denote by $\mathbf{m}^i$ the solution to Problem 1_i. Then the difference $\overline{\mathbf{m}} := \mathbf{m}^1 - \mathbf{m}^2$ enjoys*

$$\operatorname*{ess\,sup}_{t\in I}(\|\overline{\mathbf{m}}\|_{W^{2,2}} + \|\partial_t \overline{\mathbf{m}}\|_2) + \left[\int_0^T \|\partial_t \overline{\mathbf{m}}\|^2_{W^{1,2}}\right]^{\frac{1}{2}} \le C(\|\beta^1 - \beta^2\|_\infty).$$

Appendix C. Stability of the sensitivity equations

The α-stability. Take two functions α^1, α^2 and suppose $\mathbf{m}^1$, $\mathbf{m}^2$ are the corresponding solutions to Problem 1 with the same initial condition (IC), respectively. Suppose $\mathbf{v}_1^i$ are solutions to the first sensitivity equation (7) where function $\alpha = \alpha^i$, $i = 1, 2$ with the same IC. Then the difference $\overline{\mathbf{v}}_1 := \mathbf{v}_1^1 - \mathbf{v}_1^2$ obeys

$$\begin{aligned}\partial_t \overline{\mathbf{v}}_1 - \alpha^2 \Delta \overline{\mathbf{v}}_1 - \alpha^2 R(\mathbf{m}^2, \overline{\mathbf{v}}_1) - \beta S(\mathbf{m}^2, \overline{\mathbf{v}}_1) &= (\alpha^1 - \alpha^2)\Big[R(\mathbf{m}^1, \mathbf{v}_1^1) + \Delta \mathbf{v}_1^1 + \mu_1 P_1(\mathbf{m}^1)\Big] \\ &\quad + \alpha^2\Big[R(\mathbf{m}^1, \mathbf{v}_1^1) - R(\mathbf{m}^2, \mathbf{v}_1^1)\Big] + \beta\Big[S(\mathbf{m}^1, \mathbf{v}_1^1) - S(\mathbf{m}^2, \mathbf{v}_1^1)\Big] + \alpha^2\mu_1\Big[P_1(\mathbf{m}^1) - P_1(\mathbf{m}^2)\Big].\end{aligned} \tag{24}$$

We have again obtained a system similar to (7) with the same principal operator and a slightly different right-hand side consisting of four terms denoted by A_1, A_2, A_3, A_4, respectively. Suppose the assumptions of Theorem 1 are satisfied for $k = 4$. Then, using the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ we can estimate $\mathbf{m}^i$, $\nabla \mathbf{m}^i$ and $\Delta \mathbf{m}^i$ in $L^\infty(\Omega_T)$ by some constant C

for $i = 1, 2$. Further from Theorem 3 we have directly boundedness of $\mathbf{v}_1^i$ in $L^\infty(I, W^{2,2}(\Omega))$ again for $i = 1, 2$. Using these preliminaries we can estimate terms A_1, A_2, A_3, A_4 in the following way. For an arbitrary t we can conclude that

$$\begin{aligned}\|A_1\|_2 &= \left\|(\alpha^1 - \alpha^2)\left[R(\mathbf{m}^1, \mathbf{v}_1^1) + \Delta\mathbf{v}_1^1 + \mu_1 P_1(\mathbf{m}^1)\right]\right\|_2 \\ &\le C\|\alpha^1 - \alpha^2\|_\infty[\|\mathbf{m}^1\|_\infty + \|\nabla\mathbf{m}^1\|_\infty + \|\Delta\mathbf{m}^1\|_\infty](\|\mathbf{v}_1^1\|_{W^{2,2}} + \|\mu_1\|_\infty) \le C\|\alpha^1 - \alpha^2\|_\infty\|\mu_1\|_\infty,\end{aligned}$$

using Theorem 3 at the end. For term A_2 we can write

$$\begin{aligned}\|A_2\|_2 &= \left\|\alpha^2\left[R(\mathbf{m}^1, \mathbf{v}_1^1) - R(\mathbf{m}^2, \mathbf{v}_1^1)\right]\right\|_2 \\ &\le \|\alpha^2\|_\infty\Big[2\|\langle\nabla\mathbf{v}_1^1, \nabla\overline{\mathbf{m}}\rangle\mathbf{m}^1\|_2 + 2\|\langle\nabla\mathbf{v}_1^1, \nabla\mathbf{m}^2\rangle\overline{\mathbf{m}}\|_2 + \|\langle\nabla\overline{\mathbf{m}}, \nabla\mathbf{m}^1\rangle\mathbf{v}_1^1\|_2 + \|\langle\nabla\mathbf{m}^2, \overline{\mathbf{m}}\rangle\mathbf{v}_1^1\|_2\Big] \\ &\le C\Big[\|\nabla\mathbf{v}_1^1\|_4\|\nabla\overline{\mathbf{m}}\|_4 + \|\nabla\mathbf{v}_1^1\|_4\|\overline{\mathbf{m}}\|_4 + \|\nabla\overline{\mathbf{m}}\|_4\|\mathbf{v}_1^1\|_4 + \|\overline{\mathbf{m}}\|_4\|\mathbf{v}_1^1\|_4\Big].\end{aligned}$$

Using the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ we go further, and again using Theorems 3 and 4, we arrive at

$$\|A_2\|_2 \le C\|\mathbf{v}_1^1\|_{W^{2,2}}\|\overline{\mathbf{m}}\|_{W^{2,2}} \le C\|\mu_1\|_\infty\|\alpha^1 - \alpha^2\|_\infty.$$

For term A_3 we have a similar result, again using Theorems 3 and 4 at the end

$$\begin{aligned}\|A_3\|_2 &= \left\|\beta\left[S(\mathbf{m}^1, \mathbf{v}_1^1) - S(\mathbf{m}^2, \mathbf{v}_1^1)\right]\right\|_2 \le \|\beta\|_\infty\Big[\|\overline{\mathbf{m}} \times \Delta\mathbf{v}_1^1\|_2 + \|\mathbf{v}_1^1 \times \Delta\overline{\mathbf{m}}\|_2\Big] \\ &\le C\Big[\|\overline{\mathbf{m}}\|_\infty\|\Delta\mathbf{v}_1^1\|_2 + \|\mathbf{v}_1^1\|_\infty\|\Delta\overline{\mathbf{m}}\|_2\Big] \le C\|\mathbf{v}_1^1\|_{W^{2,2}}\|\overline{\mathbf{m}}\|_{W^{2,2}} \le C\|\mu_1\|_\infty\|\alpha^1 - \alpha^2\|_\infty.\end{aligned}$$

Finally for term A_4 we get

$$\begin{aligned}\|A_4\|_2 &= \left\|\alpha^2\mu_1\left[P_1(\mathbf{m}^1) - P_1(\mathbf{m}^2)\right]\right\|_2 \\ &\le C\|\mu_1\|_\infty\Big[\|\Delta\overline{\mathbf{m}}\|_2 + \|\langle\nabla\overline{\mathbf{m}}, \nabla\mathbf{m}^1 + \nabla\mathbf{m}^2\rangle\mathbf{m}^1\|_2 + \||\nabla\mathbf{m}^2|^2\overline{\mathbf{m}}\|_2\Big] \\ &\le C\|\mu_1\|_\infty\|\overline{\mathbf{m}}\|_{W^{2,2}} \le C\|\mu_1\|_\infty\|\alpha^1 - \alpha^2\|_\infty,\end{aligned}$$

where we have used Theorem 4 at the end.

Summarizing the previous four estimates we succeed in estimating the right-hand side of (24) in the norm of the space $L^\infty(I, L^2(\Omega))$ by $C\|\mu_1\|_\infty\|\alpha^1 - \alpha^2\|_\infty$. This again allows for the application of [13, Theorem 5, Section 7.1] for the solution to (24). We state this result in the following theorem.

Theorem 6. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. For an arbitrary $\alpha^i \in L^\infty(\Omega)$, $i = 1, 2$ consider Problem 1_i derived from Problem 1 simply by replacing α with α^i and denote by $\mathbf{m}^i$ their solutions. Further consider two sensitivity equations with respect to α corresponding to Problem 1_i and denote by $\mathbf{v}_1^i$ their solutions, $i = 1, 2$. Then the difference $\overline{\mathbf{v}_1} := \mathbf{v}_1^1 - \mathbf{v}_1^2$ enjoys*

$$\operatorname*{ess\,sup}_{t\in I}\|\overline{\mathbf{v}_1}\|_{W^{1,2}} + \left[\int_0^T \|\overline{\mathbf{v}_1}\|_{W^{2,2}}^2\right]^{\frac{1}{2}} + \left[\int_0^T \|\partial_t\overline{\mathbf{v}_1}\|_2^2\right]^{\frac{1}{2}} \le C\|\mu_1\|_\infty\|\alpha^1 - \alpha^2\|_\infty.$$

The β-stability. The stability of the sensitivity equation on β can be obtained repeating all the steps from the section. We present the resulting theorem only.

Theorem 7. *Let the assumptions of Theorem 1 be satisfied for $k = 4$. For an arbitrary $\beta^i \in L^\infty(\Omega)$, $i = 1, 2$ consider Problem 1_i derived from Problem 1 simply by replacing β with β^i and denote by $\mathbf{m}^i$ their solutions. Further consider two sensitivity equations with respect to β corresponding to Problem 1_i and denote by $\mathbf{v}_2^i$ their solutions, $i = 1, 2$. Then the difference $\overline{\mathbf{v}_2} := \mathbf{v}_2^1 - \mathbf{v}_2^2$ enjoys*

$$\operatorname*{ess\,sup}_{t\in I}\|\overline{\mathbf{v}_2}\|_{W^{1,2}} + \left[\int_0^T \|\overline{\mathbf{v}_2}\|_{W^{2,2}}^2\right]^{\frac{1}{2}} + \left[\int_0^T \|\partial_t\overline{\mathbf{v}_2}\|_2^2\right]^{\frac{1}{2}} \le C\|\mu_2\|_\infty\|\beta^1 - \beta^2\|_\infty.$$

References

[1] I. Cimrák, V. Melicher, The Landau–Lifshitz model for shape optimization of MRAM memories, PAMM 6 (1) (2006) 23–26.
[2] I. Cimrák, V. Melicher, Sensitivity analysis framework for micromagnetism with application to optimal shape design of MRAM memories, Inverse Problems 23 (2007) 563–588.
[3] G. Bertotti, Hysteresis in Magnetism, Academic Press, 1998.
[4] I. Cimrák, A survey on the numerics and computations for the Landau–Lifshitz equation of micromagnetism, Arch. Comput. Methods Engrg. 15 (3) (2008) 277–309.
[5] L'. Baňas, A. Prohl, M. Slodička, Modeling of thermally assisted magnetodynamic, SIAM J. Numer. Anal. 47 (1) (2008) 551–574.
[6] A. Prohl, Computational Micromagnetism, in: Advances in Numerical Mathematics, B.G. Teubner, Stuttgart, 2001.

[7] G. Carbou, P. Fabrie, Regular solutions for Landau–Lifshitz equation in a bounded domain, Differential Integral Equations 14 (2) (2001) 213–229.
[8] M.M. Vainberg, Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations, John Wiley & Sons, New York, Toronto, 1973.
[9] V.A. Morozov, On the solution of functional equations by the method of regularization, Doklady Akad. Nauk SSSR 7 (1966) 414–417 (in Russian).
[10] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, in: Mathematics and its Applications, vol. 375, Kluwer Academic Publishers, Dordrecht, 1996.
[11] I. Cimrák, Error estimates for a semi-implicit numerical scheme solving the Landau–Lifshitz equation with an exchange field, IMA J. Numer. Anal. 25 (2005) 611–634.
[12] O. Pironneau, F. Hecht, A. Le Hyaric, Freefem++ package. http://www.freefem.org.
[13] L. Evans, Partial Differential Equations, in: Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.